Government 2005: Formal Political Theory I

Lecture 11

Instructor: Tommaso Nannicini
Teaching Fellow: Jeremy Bowles

Harvard University

November 9, 2017

Overview

* **Today's lecture**
- Dynamic games of incomplete information
  - Definitions
  - Perfect Bayesian equilibrium (PBE)
  - Examples
- Signaling games
  - Signaling in the labor market (Spence 1973)
  - PBE refinements (domination-based & intuitive criterion)

* **What's next (lecture 12)**
- Political agency and accountability (Besley 2006)
- Cheap talk games (Crawford and Sobel 1982)

Dynamic games of incomplete information

- Bayesian environment where players have priors on information they do not possess
- And they revise beliefs (forming posteriors) as the sequential interaction with the other players unfolds
- Private information can be on what some players are (**hidden information**) or on what some players do (**hidden action**)
- Useful classification in contract theory:
  - **Adverse selection/screening models**. Uninformed players (about characteristics of informed players) move first
    - E.g., insurance market, voting politicians of unobserved quality
  - **Moral hazard models**. Uninformed players (about actions of informed players) move first
    - E.g., unemployment benefits, re-voting for incumbent
  - **Signaling models**. Informed players (about their own characteristics) move first
    - E.g., education in the labor market, entry deterrence in elections, informative lobbying

Perfect Bayesian equilibrium

- New solution concept: Perfect Bayesian equilibrium

	Complete information	Incomplete information
Static	Nash	Bayesian Nash
Dynamic	Subgame-perfect	**Perfect Bayesian**

- Key ingredients:
  - Sequential rationality
  - Bayesian updating
- In hidden-information games, we can characterize:
  - Pooling PBE
  - Separating PBE
  - But these are not solution concepts, just characterization of the equilibria that may emerge with PBE concept

Perfect Bayesian equilibrium (contd.)

We define PBE as a strategy-belief pair that satisfies the following four requirements:

A At each information set, the player with the move must have a *belief* about which node in the information set has been reached by the play of the game

B Given their beliefs, the players' strategies must be *sequentially rational*

C At information sets *on the equilibrium path*, beliefs are determined by Bayes' rule and the players' equilibrium strategies

D At information sets *off the equilibrium path*, beliefs are determined by Bayes' rule and the players' equilibrium strategies *whenever possible*

Perfect Bayesian equilibrium (contd.)

Let's define these requirements a little bit more formally:

A At each $h \in H$, define $k(h)$ as the player with the move; she must have beliefs on each node x of h s.t. $\mu(x) \in [0,1]$ and $\sum_{x \in h} \mu(x) = 1$

B A strategy profile $s = (s_1, ..., s_I)$ is sequentially rational at h given μ if $\forall \tilde{s}_{k(h)} \in S_{k(h)}$:

$$E[u_{k(h)}|h, \mu, s_{k(h)}, s_{-k(h)}] \geq E[u_{k(h)}|h, \mu, \tilde{s}_{k(h)}, s_{-k(h)}]$$

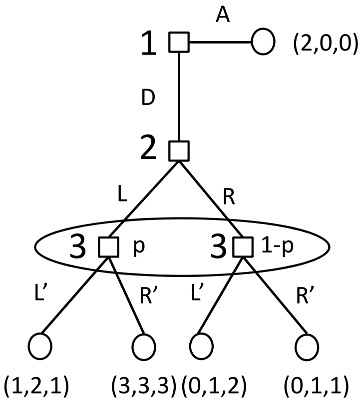
C At each h with $Prob[h|s] > 0$, $\forall x \in h$, beliefs are given by:

$$\mu(x) = \frac{Prob[x|s]}{Prob[h|s]}$$

D At each h with $Prob[h|s] = 0$, beliefs are determined by Bayes' rule and the players' equilibrium strategies *whenever possible* (we'll specify this game by game)

A first example

- To illustrate these requirements, consider the following three-player game



- p is the belief that player 3 has about player 2 playing L

A first example (contd.)

- The game has a unique subgame, which begins at player 2's singleton information set

	L'	R'
L	(2,1)	(**3**,**3**)
R	(1,2)	(1,1)

- The unique NE of this subgame between players 2 and 3 is (L, R'), so the unique SPNE of the entire game is (D, L, R')
- The game has multiple NE, however
  - For example, (A, L, L') is a NE—no player has incentive to deviate unilaterally

A first example (contd.)

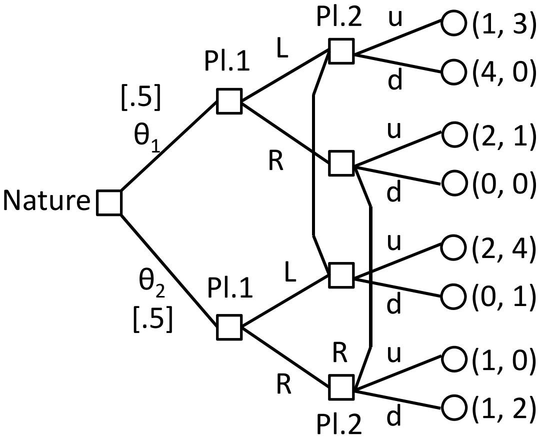
- What about the **perfect Bayesian equilibrium** of this game?
- When player 3 has to move, she must have a belief about whether she is at the left or right node of her information set (requirement A)—let p be the probability she believes she is at the left node
- Consider the strategy-belief pair (D, L, R') and $p = 1$
- It is straightforward to check that they satisfy requirements A-B-C above (weak PBE)
- They also trivially satisfy requirement D, since there is no information set off this equilibrium path
- So, the strategy-belief pair (D, L, R') and $p = 1$ is a PBE

A first example (contd.)

- Consider the strategy-belief pair (A, L, L') and $p = 0$
- The strategies and beliefs satisfy requirements A-B-C above (weak PBE): Along the equilibrium path, player 3's information set is never reached, so requirement 3 places no restrictions on player 3's beliefs
- Requirement D, however, does restrict player 3's belief at her information set
  - If strategies are given by (A, L, L'), then player 3 *cannot* have the belief $p = 0$
  - This belief is inconsistent with player 2's strategy
  - If player 2 is playing L, then player 3's belief *must* be $p = 1$ to satisfy requirement D
- So, the strategy-belief pair (A, L, L') and $p = 0$ is not a PBE

A second example

- A simple example of signaling game, with 2 types, 2 possible messages, and 2 possible actions
- The sender's type is either θ_1 or θ_2, the message is either R or L, and the receiver's response is either u or d



A second example (contd.)

- Let p be the receiver's belief that $\theta = \theta_1$ given that the sender's message is L, and let q be the receiver's belief that $\theta = \theta_1$ given that the sender's message is R
- Define $m(.)$ as the message of player 1 and $r(.)$ as the response of player 2. A pure-strategy PBE is a triple $[(m(\theta_1), m(\theta_2)), (r(L), r(D)), (p, q)]$ satisfying requirements A-B-C above
- Requirement D is vacuous in signaling games (the sender's strategy does not restrict the receiver's beliefs off the equilibrium path; that's why we'll discuss PBE refinements in signaling games below)
- In this game there are four types of pure-strategy PBE:
  - Pooling equilibria with $m(\theta_1) = m(\theta_2) = L$
  - Pooling equilibria with $m(\theta_1) = m(\theta_2) = R$
  - Separating equilibria with $m(\theta_1) = L$ and $m(\theta_2) = R$
  - Separating equilibria with $m(\theta_1) = R$ and $m(\theta_2) = L$

A second example (contd.)

Pooling on L

- If player 1 picks (L, L), $p = 1/2$ on the equilibrium path
- Best response of player 2 is $r(L) = u$
- What about $r(R)$ (off the equilibrium path)?
- This must be d, because if it were u, θ_1 would have incentive to deviate from L to R
- Hence, for the equilibrium to be pooling on L, we must have (L, L), (u, d), and $p = 1/2$
- But what about q? What values the off-equilibrium belief must have to sustain $r(R) = d$?
- It's easy to see: $q + 0 \leq 0 + 2(1 - q) \Rightarrow q \leq 2/3$
- Any $[(L, L), (u, d), p = 1/2, q \leq 2/3]$ is a pooling PBE

A second example (contd.)

Pooling on R

- If player 1 picks (R, R), $q = 1/2$ on the equilibrium path
- Best response of player 2 is $r(R) = d$
- But in this case θ_1 would get zero and have an incentive to deviate to L
- As a result, there cannot be a PBE with pooling on R

A second example (contd.)

Separating with θ_1 playing L

- Both information sets of player 2 are on the equilibrium path
- Therefore, her beliefs must be: $p = 1$ and $q = 0$
- Her best responses are $r(L) = u$ and $r(R) = d$, that is, (u, d)
  But in this case θ_2 has incentive to deviate to L (as $2 > 1$)
- As a result, there cannot be a separating PBE in which $m(\theta_1) = L$ and $m(\theta_2) = R$

A second example (contd.)

Separating with θ_1 playing R

- Both information sets of player 2 are on the equilibrium path
- Therefore, her beliefs must be: $p = 0$ and $q = 1$
- Her best responses are $r(L) = u$ and $r(R) = u$, that is, (u, u)
  Both θ_1 and θ_2 get payoff of 2 and have no incentive
- As a result, $[(R, L), (u, u), p = 0, q = 1]$ is a separating PBE

Job market signaling

- Consider the following game involving two firms trying to hire a worker of unknown productivity (it's easy to generalize this to a population of workers)
- The worker can have either high or low productivity, but firms do not observe the worker's type. The high-productivity worker can buy education as a signal
- The timing of the game is as follows

1. Nature determines the worker's productive ability, $\theta \in \{\theta_L, \theta_H\}$, with $\theta_L < \theta_H$ and $Prob[\theta = \theta_H] = \lambda \in (0, 1)$
2. The worker learns θ and chooses an education level, $e \geq 0$
3. The firms observe e (but not θ) and simultaneously make wage offers
4. The worker accepts the higher wage, w (flipping a fair coin in case of a tie)

Job market signaling (contd.)

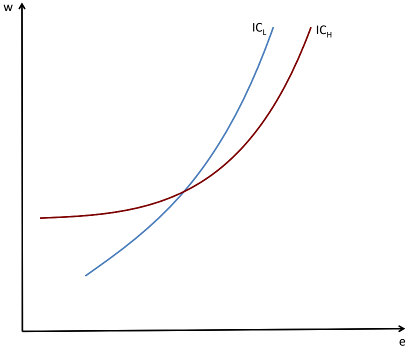
- The payoff to the firm that hires the worker is: $\pi = \theta - w$
- The payoff to the other firm is: $\pi = 0$
- The worker's payoff is: $u(w, e|\theta) = w - c(\theta, e)$
- Assume that $c(\theta, e)$ for a given θ is increasing and convex in e: $\partial c/\partial e > 0$, $\partial^2 c/\partial e^2 > 0$
- Assume that both the cost and the marginal cost of e are decreasing in θ: $\partial c/\partial \theta < 0$, $\partial^2 c/\partial e \partial \theta < 0$
- Assume that the worker's reservation utility is zero (so we don't have to bother about her individual rationality constraint): $r(\theta_L) = r(\theta_H) = 0$

- There are **tons of different perfect Bayesian equilibria**
- Let's try to characterize them

Job market signaling (contd.)

Single-crossing property

- The assumption we made on the cross partial, $\partial^2 c/\partial e \partial \theta < 0$, has important implications for the game
- It means that indifference curves in (e, w) for the high vs low type cross just once, and when they do the IC of θ_L is steeper[1]



[1] Note that any IC is given by $\overline{u} = w - c(e, \theta)$ and thus their slope is given by $\partial w/\partial e = \partial c/\partial e$, which is lower for θ_H than θ_L

Job market signaling (contd.)

Firms earn zero profits in equilibrium

- Worker's strategy is $e(\theta)$
- Firms' strategy is $w(e)$
- Bertrand competition between the firms drives profits to zero
- As education has no effect on productivity:[2]
  - $w(e) = E[\theta|e]$, that is,
  - $w(e) = \mu(e)\theta_H + (1 - \mu(e))\theta_L$,
  - where $\mu(e)$ is the (common) firms' posterior: $\mu(e) = Prob[\theta = \theta_H|e]$
- This also means that: $\theta_L \leq w(e) \leq \theta_H$ for any e
- Now, let's look for PBE that are **separating equilibria (SE)**, that is, PBE where the optimal strategies are different for the two worker's types: $e^*(\theta_L) \neq e^*(\theta_H)$

[2]Note that this is not the case in Gibbons, but nothing changes about the nature of the game, results are just starker here

Job market signaling (contd.)

Separating equilibria

- **Result 1.** In any SE we must have that:
  - $\mu(e^*(\theta_L)) = 0$
  - $\mu(e^*(\theta_H)) = 1$
  - $w^*(e^*(\theta_L)) = \theta_L$
  - $w^*(e^*(\theta_H)) = \theta_H$
  - Indeed, on the equilibrium path, beliefs must be correctly derived from equilibrium strategies using Bayes' rule
  - As firms can disentangle the low type from the high type, they end up offering her productivity to each type

- **Result 2.** In any SE we must have that:
  - $e^*(\theta_L) = 0$
  - Indeed, there is no point for the low type to waste resources in education if she gets the low wage anyway

Job market signaling (contd.)

Separating equilibria (contd.)

- Consider the SE with $e^*(\theta_L) = 0$ and $e^*(\theta_H) = e'$ in **Fig.1**: Is it a PBE?
- The answer is no because the pair $(e = e', w = \theta_H)$ is above the IC_L associated with $e = 0$ and thus the low type would have an incentive to mimic the high type by acquiring e'
- In other words, the **incentive compatibility constraint** of the low type (ICC_L) is violated there:

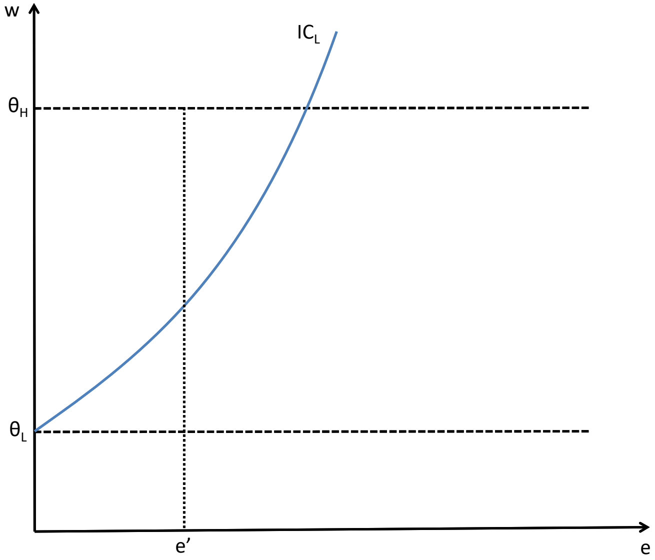
$$w^*(0) - c(\theta_L, 0) < w^*(e') - c(\theta_L, e')$$

- Note that in general the ICC_L is given by:

$$w^*(e^*(\theta_L)) - c(\theta_L, e^*(\theta_L)) \geq w^*(e^*(\theta_H)) - c(\theta_L, e^*(\theta_H))$$

Job market signaling (contd.)

Figure 1: SE that is not a PBE



Job market signaling (contd.)

Separating equilibria (contd.)

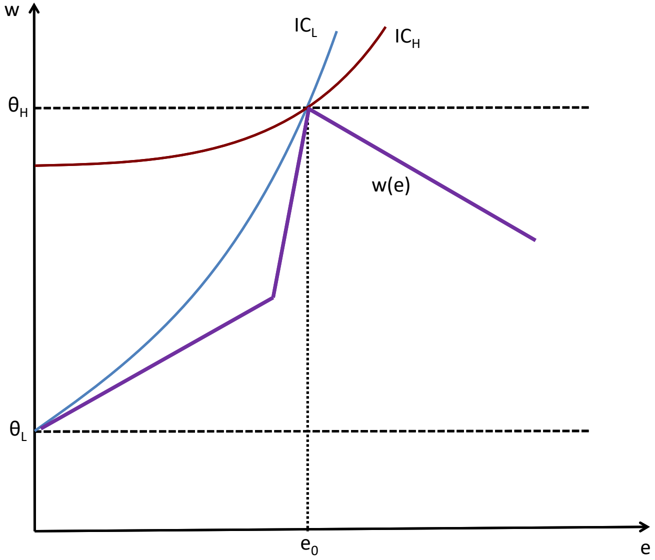
- Consider the SE with $e^*(\theta_L) = 0$ and $e^*(\theta_H) = e_1$ in **Fig.2**: Is it a PBE?
- The answer is yes because the pair $(e = e_1, w = \theta_H)$ is not above the IC_L associated with $e = 0$ and thus the low type has no incentive to mimic the high type by acquiring e_1
- In other words, the ICC_L is met there:

$$w^*(0) - c(\theta_L, 0) \geq w^*(e_1) - c(\theta_L, e_1)$$

- But what about the high type? Her IC_H in the SE is clearly above the one associated with $e = 0$
- Moreover, there are many firms' posterior beliefs and associated $w(e)$ that can induce the high type not to acquire a lower $e > 0$, as this would result in the possibility to be confused with the low type and get a lower wage
- The purple curve $w(e)$ in Fig.2 is one of the many wage curves/beliefs that can sustain this SE

Job market signaling (contd.)

Figure 2: SE that is a PBE



Job market signaling (contd.)

Welfare analysis of separating equilibria

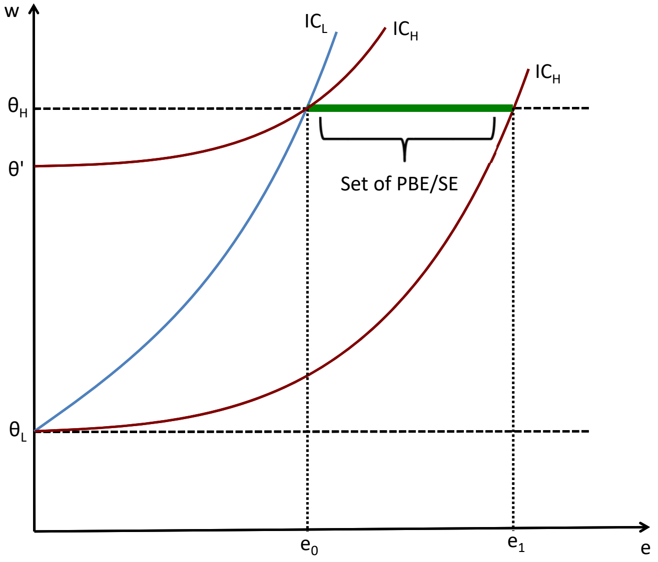
- By the same reasoning we can find $w(e)$ that sustain many other SE, namely, all $e^*(\theta_H) \in [e_0, e_1]$ in **Fig.3** (green interval)
- Education levels above e_1 are not sustainable because the ICC_H is violated there (i.e., the high type would prefer the low wage with no education)
- Note that in general the ICC_H is given by:

$$w^*(e^*(\theta_H)) - c(\theta_H, e^*(\theta_H)) \geq w^*(e^*(\theta_L)) - c(\theta_H, e^*(\theta_L))$$

- There's an infinite set of SE as the PBE concept imposes almost no restriction on the beliefs off the equilibrium path
- These SE can be Pareto ranked: Firms always gets zero and the low type θ_L, but the high type is better off with e_0
- If $E[\theta] > \theta'$ in Fig.3, we have the paradox that education makes everybody worse off, as also the high type would prefer the situation with incomplete information but no signaling

Job market signaling (contd.)

Figure 3: Set of all SE that are PBE



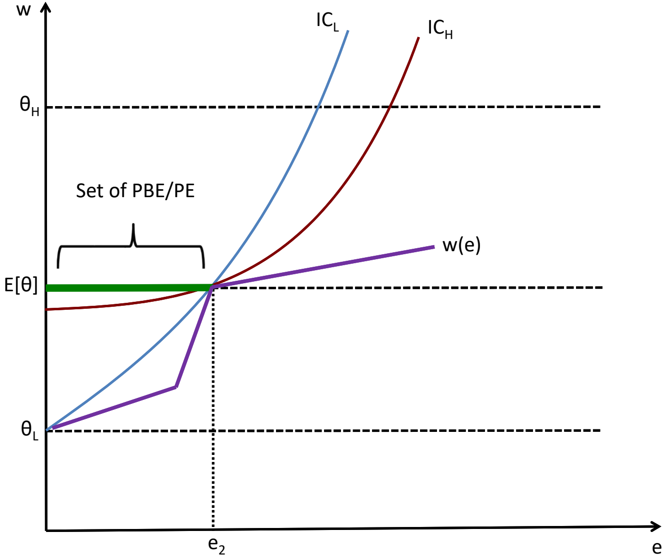
Job market signaling (contd.)

Pooling equilibria

- We now turn to PBE that are **pooling equilibria (PE)**, that is, PBE where the optimal strategies are identical for the two worker's types: $e^*(\theta_L) = e^*(\theta_H) = e^*$
- In this situation, firms posterior is equal to the prior and we must have: $w^*(e^*) = \lambda\theta_H + (1-\lambda)\theta_L = E[\theta]$
- Consider the PE $e^* = e_2$ in **Fig.4**:
  - The ICC_L is (barely) met (note that when the ICC_L is met, the ICC_H is trivially met in PE)
  - And, given the (purple) $w(e)$ (or any wage/beliefs curve below IC_H off equilibrium), also the high type prefers the PE instead of getting more education
  - Hence, this is a PBE
- This holds for all $e^* \in [0, e_2]$, as ICC_L violated above e_2; the green interval in Fig.4 is the (infinite) set of PBE/PE
- Equilibrium with $e^* = 0$ Pareto dominates all the others

Job market signaling (contd.)

Figure 4: Set of all PE that are PBE



PBE refinements

- In the above discussion, Pareto-dominated PBE are sustained by off-equilibrium firms' beliefs
- Indeed, the multiplicity of equilibria arises from great freedom we have in choosing beliefs off the equilibrium path with the PBE concept
- Are all of these beliefs really reasonable?
- Now, we'll impose stricter definitions of "being reasonable" and restrict the set of PBE in this way
- In particular, we'll consider two PBE refinements:
  - **Domination-based refinements of beliefs**
  - **Intuitive criterion**

Domination-based refinements

- Consider $e^*(\theta_H) = e_1$ in Fig.3. This SE sustained by belief that $\mu(e) < 1$ for $e < e_1$
- Are these beliefs reasonable?
- Indeed, you can never convince θ_L to get $e \in (e_0, e_1]$, regardless of what firms believe of her type as a result
- We should rule out these beliefs, based on dominated strategies by θ_L
- Action e is dominated for θ_i if there's e' s.t.

$$Min_{w(e')} u(e', w(e'), \theta_i) > Max_{w(e)} u(e, w(e), \theta_i)$$

Domination-based refinements (contd.)

- Off the equilibrium path, after observing e, firms should believe $Prob[\theta = \theta_i] = 0$ if e is dominated for θ_i (if possible, i.e., if e is not dominated for all types)
- With this refinement, any $e > e_0$ is dominated by zero for θ_L and then we should have $\mu(\theta_H | e > e_0) = 1$
- No SE with $e > e_0$ survives this refinement
- Moreover, we should also rule out any PE in which θ_H is worse off with respect to $(w = \theta_H, e = e_0)$ (try to draw the new set of PE sustainable with this refinement)
- This also implies that if $E[\theta] < \theta'$ in Fig.3, no PE survives this refinement and we have a unique SE
- But if $E[\theta] \geq \theta'$, the set of (refined) PBE is made up of the SE with $e^*(\theta_H) = e_0$ in Fig.3 plus a subset of the green interval of PE in Fig.4

Intuitive criterion

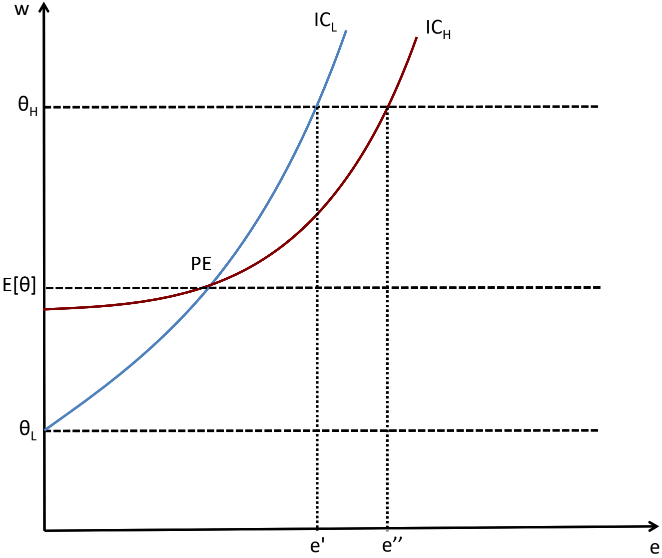
- Let's now consider an even stricter PBE refinement
- Definition. Action e is **equilibrium dominated** if θ_i's equilibrium payoff is greater than θ_i's highest payoff from e:

$$u^*(\theta_i) > Max_{w(e)} u(e, w(e), \theta_i)$$

- **Intuitive criterion**. Off the equilibrium path, after observing e, firms should believe $Prob[\theta = \theta_i] = 0$ if e is equilibrium dominated for θ_i (if possible)
- Consider the PE in **Fig.5**:
  - To sustain it, $\mu(\theta_H|e) < 1$ if $e \in (e', e'')$, otherwise θ_H deviates from pooling
  - But if firms observe $e \in (e', e'')$ while expecting PE, by the intuitive criterion, they should place zero probability to the fact that the worker is θ_L
  - As a result, the intuitive criterion kills all PE
  - The SE with $e^*(\theta_H) = e_0$ is the unique equilibrium satisfying the intuitive criterion

Intuitive criterion (contd.)

Figure 5: Intuitive criterion kills all PE



Where are we?

- We have (briefly) studied **dynamic games of incomplete information** (in particular the sub-set of **signaling games**)
- References:[3]
  - Lecture slides $\rightarrow$ 11 (final folder)
  - Osborne $\rightarrow$ chapter 10
  - Gibbons $\rightarrow$ chapter 4
- In the next class, we'll extend this discussion by:
  - Studying a game of political accountability
  - Discussing cheap talk games (with costless signals)

[3] Those of you interested in a more advanced (game-theoretic) treatment of contract theory may want to have a look at the graduate-level textbook of Bernard Salanié, *The Economics of Contracts*