Government 2005
Formal Political Theory I
Lecture 2

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Lecture 2: Overview

- Nash equilibrium: Discussion
- Nash equilibrium: Applications
  - Collective action and public goods
  - Collective action and interest groups
  - Strategic substitutes vs strategic complements
- Mixed strategies
- Flashback I: Dominated strategies
- Flashback II: Rationalizable strategies

Nash reloaded

- Remember our definition of Nash equilibrium:

A strategy profile $s = (s_1, ..., s_I)$ constitutes a Nash equilibrium of the game $\Gamma_N = \langle I, S_i, u_i(.)\rangle$ if for every player $i = 1, ..., I$:

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \forall s_i' \in S_i$$

- It means that each player's choice is **best response** to strategies actually played by the others
- More than rationality plus common knowledge (see rationalizability below): here, conjectures/beliefs about what other players will do must be correct in equilibrium
- Does this make sense?
- A. Lincoln: "*You can fool all of the people some of the time, and some of the people all of the time, but you can't fool all of the people all of the time*"

Nash reloaded (contd.)

- Define the **best-response correspondence** ($b_i : S_{-i} \rightarrow S_i$) for player i in game Γ_N as:

$$b_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \forall s_i' \in S_i\}$$

- We can re-define Nash equilibrium of game Γ_N as:

Strategy profile $(s_1, ..., s_I)$ such that $s_i \in b_i(s_{-i})$ for every i

- We can think of it as:
  - Rest point of some dynamic adjustment
  - Focal point (Schelling 1960)
  - Stable social convention
  - Self-enforcing agreement after non-binding communication

Nash reloaded (contd.)

- Do we have any evidence that people play Nash? We'll come back to lab evidence in the last class. A few points for now:
  - Correct test must look at both rationality and correct beliefs
  - Correct test must induce assumed preferences
  - Correct test must replicate steady-state situation
- Let's go back to our simple games and solve them by identifying Nash equilibria (if any):
  - Prisoner's dilemma
  - Game of chicken
  - Assurance dilemma (or "stag hunt")
  - Generals' dilemma (or "matching pennies")
- These (simple) strategic games give us important insights to examine collective action (e.g., provision of public goods, organization of trade unions, international externalities, riots or voting participation)

Collective action

- In prisoner's dilemma types of situation:
  (a) no one finds it profitable to cooperate alone
  (b) value of the others' cooperation when I free-ride exceeds value of joint cooperation minus cost to cooperate for me
  - Because of (a): no deviation from non-cooperative eq.
  - Because of (b): incentive to deviate from cooperative eq.
- In chicken's game types of situation:
  - No (a)—each player prefers lonely cooperation than joint non-cooperation—but (b) still holds
  - "Privileged groups" (Olson 1965)
  - We observe free-riding in equilibrium
  - Incentive to pre-commitment into non-cooperation
- In assurance dilemma types of situation:
  - No (b) but (a) still holds
  - Cooperative equilibrium becomes stable
  - Heterogeneous agents & thresholds models (Granovetter 1978)

Contributing to a public good

- Let's solve this game together in class:
  - Each of n people chooses whether or not to contribute a fixed amount toward the provision of a public good. The good is provided iff at least k people contribute, where $2 \leq k \leq n$; if it is not provided, contributions are not refunded. Each person ranks outcomes from best to worst as follows: (i) any outcome in which the good is provided and she does not contribute, (ii) any outcome in which the good is provided and she contributes, (iii) any outcome in which the good is not provided and she does not contribute, (iv) any outcome in which the good is not provided and she contributes

Sketch of the solution

- Define k^* as number of contributing players in equilibrium
- $k^* > k$: no Nash (contributors have incentive to deviate)
- $k^* = k$: NE
- $0 < k^* < k$: no Nash (contributors have incentive to deviate)
- $k^* = 0$: NE
- Therefore, set of Nash equilibria: $\{$ k players contribute and (n-k) don't $\} \cup \{$ nobody contributes $\}$

Voters' turnout

- Let's solve this game together in class:
  - Two candidates, A and B, compete in an election. Of the n citizens, k support candidate A and m $(= n - k)$ support candidate B. Each citizen decides whether to vote, at a cost, for the candidate she supports, or to abstain. A citizen who abstains receives the payoff of 2 if the candidate she supports wins, 1 if this candidate ties for first place, and 0 if this candidate loses. A citizen who votes receives the payoffs $2 - c$, $1 - c$, and $-c$ in these three cases, where $0 < c < 1$
  - For $k = m = 1$, is the game the same (except for the names of the actions) as any considered so far?
  - What is the set of Nash equilibria for $k = m$?
  - What is the set of Nash equilibria for $k < m$?

Sketch of the solution

- If $k = m = 1$ we have prisoner's dilemma
  - (vote,vote) is the non-cooperative equilibrium
- If $k = m > 1$ define n_A (n_B) as number of those voting A (B)
  - $n_A = n_B = k$ is Nash as no incentive to deviate $[1 - c \rightarrow 0]$
  - $n_A = n_B < k$: no Nash as incentive to deviate from 'abstain' to 'vote' $[1 \rightarrow 2 - c]$
  - $n_A = n_B + 1$ (or $n_B = n_A + 1$): no Nash as supporters of losing candidate have incentive to deviate from 'abstain' to 'vote' $[0 \rightarrow 1 - c]$
  - $n_A \geq n_B + 2$ (or $n_B \geq n_A + 2$): no Nash as supporters of winning candidate have incentive to deviate from 'vote' to 'abstain' $[2 - c \rightarrow 2]$
  - Therefore: unique NE where everybody votes
- If $k < m$ we have no NE
  - It's enough to apply the same arguments of above to each case

Interest groups' contributions

- Let's solve this game together in class:
  - Two interest groups $I = \{1, 2\}$ seek to influence a government policy $p \in [0, 1]$. Group 1's most preferred policy is 0 and group 2's most preferred policy is 1. The government prefers $1/2$, but may be influenced by the campaign contributions. Each group chooses to contribute an amount: $s_i \in [0, 1]$. The final policy is: $p(s_1, s_2) = 1/2 - s_1 + s_2$. The groups make their choices simultaneously, and the government keeps all of the contributions to buy advertisements for the next election. The interest groups each have utility functions:

$$u_1(s_1, s_2) = -(p(s_1, s_2))^2 - s_1$$

$$u_2(s_1, s_2) = -(1 - p(s_1, s_2))^2 - s_2$$

  - What is the set of Nash equilibria (if any)?

Sketch of the solution

- The two groups maximize w.r.t. their own contribution:

$$u_1 = -(1/2 - s_1 + s_2)^2 - s_1$$

$$u_2 = -(1/2 + s_1 - s_2)^2 - s_2$$

$$FOC_1 \rightarrow 2(1/2 - s_1 + s_2) - 1 = 0$$

$$SOC_1 \rightarrow -2 < 0$$

$$FOC_2 \rightarrow 2(1/2 + s_1 - s_2) - 1 = 0$$

$$SOC_2 \rightarrow -2 < 0$$

- Therefore, infinite set of NE such that $s_1 = s_2$ as $s_1 = b_1(s_2) = s_2$ and $s_2 = b_2(s_1) = s_1$

Strategic complements

- Let's solve this game together in class:
  - Two individuals are involved in a synergistic relationship. If both individuals devote more effort to the relationship, they are both better off. For any given effort of individual j, the return to individual i's effort first increases, then decreases. Specifically, an effort level is a non-negative number, and individual i's preferences (for $i = 1, 2$) are represented by the payoff function $a_i(c + a_j - a_i)$, where a_i is i's effort level, a_j is the other individual's effort level, and $c > 0$ is a constant
  - What is the set of Nash equilibria (if any)?

Sketch of the solution

- Each agent $i = 1, 2$ maximizes w.r.t. a_i:

$$\{-a_i^2 + a_j a_i + c a_i\}$$

$$FOC_i \rightarrow -2a_i + a_j + c = 0 \rightarrow a_i = b_i(a_j) = (a_j + c)/2$$

$$SOC_i \rightarrow -2 < 0$$

- Solving system of two equations in two unknowns: $a_1 = a_2 = c$ is NE

Strategic substitutes

- Let's solve this game together in class:
  - Two countries must decide how much to invest in pollution abatement. Their strategies are levels of investment $s_1 \geq 0$ and $s_2 \geq 0$. Each country pays a cost $c(s_i) = k_i \cdot s_i$. Let $k_1 < k_2$ so that country 1 abates a given amount of pollution at a lower cost than country 2. The total amount of pollution affects citizens of both countries so that the utility of abatement is based on the total investment, $u_i(s_1, s_2) = \sqrt{s_1 + s_2}$. The payoff for each country is thus equal to:

$$\sqrt{s_1 + s_2} - (k_i \cdot s_i)$$

  - What is the set of Nash equilibria (if any)?

Sketch of the solution

$$FOC_1 \rightarrow s_1 = b_1(s_2) = (1/2k_1)^2 - s_2$$
$$FOC_2 \rightarrow s_2 = b_2(s_1) = (1/2k_2)^2 - s_1$$

- SOC verified
- No solution to system of equations as $k_1 < k_2$
- But: corner NE at $s_1 = (1/2k_1)^2$ and $s_2 = 0$
- High-cost country completely free-rides on low-cost country
- What happens if $k_1 = k_2$?

Calculating Nash equilibria in pure strategies

- We've seen different ways to calculate Nash equilibria in pure strategies
- (1) Look for best responses in (matrix) representation of a finite-strategy game
- (2) Check each strategy profile (or interval of strategy profiles) and evaluate if there are players with incentive to deviate
- (3) Unconstrained optimization to identify best response correspondences/functions through first-order and second-order conditions
  - Pay attention: these conditions are sufficient but not necessary to have Nash equilibria. There may be corner solutions or discontinuous payoff functions

Mixed strategies

- So far we have only considered Nash equilibria in pure strategies
- Some games do not have Nash equilibria in pure strategies
- We have seen this possibility in the "matching pennies" interaction

	Heads	Tails
Heads	(-1,1)	(1,-1)
Tails	(1,-1)	(-1,1)

- Each player wants to "out guess" the other

Mixed strategies (contd.)

- Best responses are in bold below

	Heads	Tails
Heads	(-1,**1**)	(**1**,-1)
Tails	(**1**,-1)	(-1,**1**)

- There is no pair such that each player's strategy is a best response to the other player's strategy
- There is no pure strategy Nash equilibrium

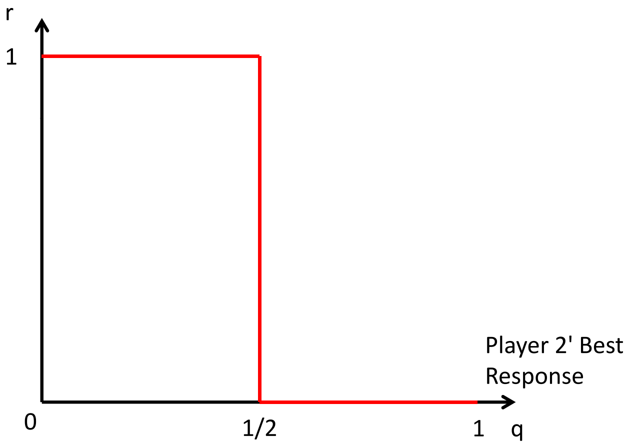
Mixed strategies (contd.)

- What is a “reasonable” prediction for this game?
- Each player should choose Heads some of the time and Tails some of the time
- We capture this formally by the concept of *mixed strategies*
- A mixed strategy is a probability distribution over pure strategies
- Let $(q, 1-q)$ be a mixed strategy for player 1
- Player 1 plays Heads with probability q, and tails with probability $1-q$
- Pure strategies are a special cases of mixed strategies: $(1, 0)$ is the strategy Heads, while $(0, 1)$ is the strategy Tails

Mixed strategies (contd.)

- Finding mixed strategies is straightforward, but a bit non-intuitive at first
- We must find an equilibrium mixed strategy for player 1 that makes player 2 *indifferent* between playing a_2 and b_2
- Suppose player 1 plays the mixed strategy $(q, 1-q)$
- Expected payoff of Heads for player 2: $q - (1-q) = 2q - 1$
- Expected payoff of Tails for player 2: $-q + (1-q) = 1 - 2q$
- If $q > \frac{1}{2}$, then player 2 should always play Heads, and if $q < \frac{1}{2}$, then he should always play Tails
- If $q = \frac{1}{2}$, then player 2 is indifferent between playing Heads and Tails

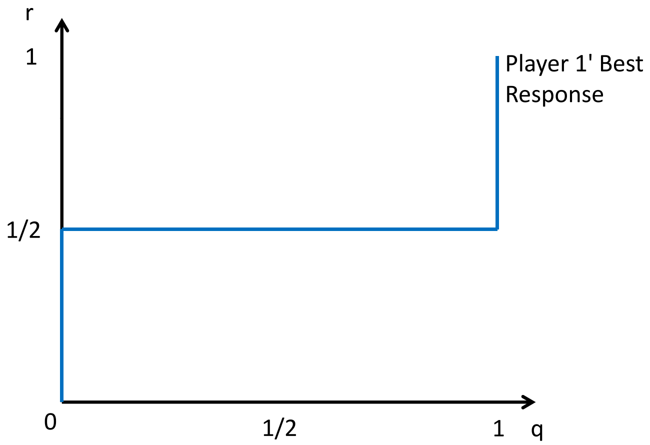
Mixed strategies (contd.)



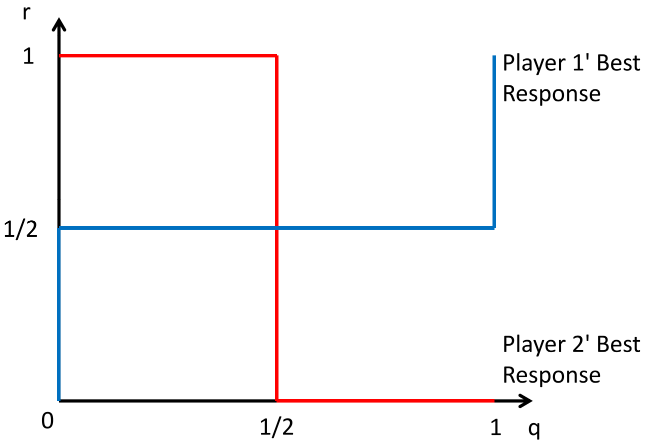
Mixed strategies (contd.)

- A similar set of calculations holds for player 1
- Let $(r, 1-r)$ be player 2's mixed strategy
- Expected payoff of Heads for player 1: $-r + (1-r) = 1 - 2r$
- Expected payoff of Tails for player 1: $r + (1-r) = 2r - 1$
- So, if $r < \frac{1}{2}$, then player 1 should always play Heads, and if $r > \frac{1}{2}$, then he should always play Tails
- If $r = \frac{1}{2}$, then he is indifferent between playing Heads and Tails
- $\mathbf{s_1} = (\frac{1}{2}, \frac{1}{2})$ and $\mathbf{s_2} = (\frac{1}{2}, \frac{1}{2})$ constitute the unique Nash equilibrium to the game

Mixed strategies (contd.)



Mixed strategies (contd.)



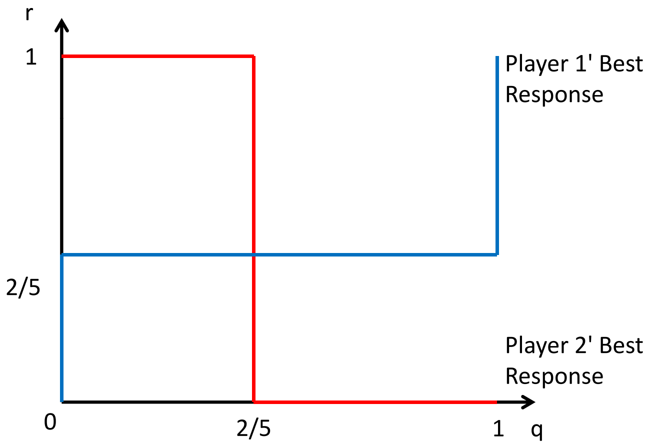
Mixed strategies (contd.)

- Suppose that player 2 (1) wins (loses) an *extra* penny from player if the coins match and are both Heads

	Heads	Tails
Heads	(-2,2)	(1,-1)
Tails	(1,-1)	(-1,1)

- Suppose player 1 plays the mixed strategy $(q, 1-q)$
- If $q = \frac{2}{5}$ then player 2 is indifferent between playing Heads and Tails
- Suppose player 2 plays the mixed strategy $(r, 1-r)$
- Player 1 is indifferent between playing Heads and Tails if $r = \frac{2}{5}$
- The strategies $\mathbf{s_1} = (\frac{2}{5}, \frac{3}{5})$ and $\mathbf{s_2} = (\frac{2}{5}, \frac{3}{5})$ constitute the unique Nash equilibrium

Mixed strategies (contd.)



Mixed strategies (contd.)

- Suppose that player 2 wins an *extra* penny from player if the coins match and are both Heads but we do not change the payoff of player 1
- Whose mixed strategies will change?
- Only the strategies of player 1 will change so that it makes player 2 indifferent
- While playing $\mathbf{s_2} = (\frac{1}{2}, \frac{1}{2})$ player 2 continues to make player 1 indifferent
- The strategies $\mathbf{s_1} = (\frac{2}{5}, \frac{3}{5})$ and $\mathbf{s_2} = (\frac{1}{2}, \frac{1}{2})$ constitute the unique Nash equilibrium

Mixed strategies (contd.)

- We can also look for mixed strategy equilibria in game with pure strategy equilibria
- Consider this game

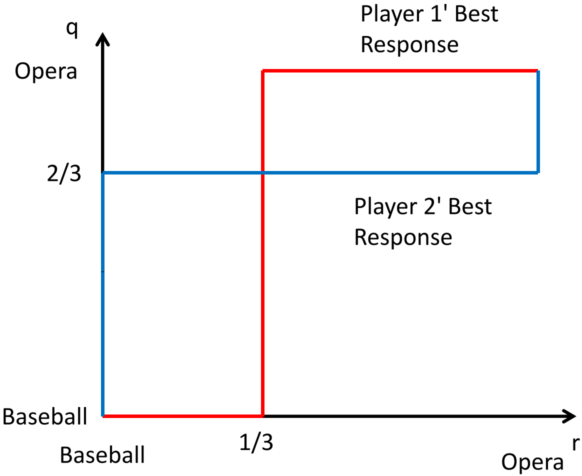
	Baseball	Opera
Baseball	(2,1)	(0,0)
Opera	(0,0)	(1,2)

- It has two pure strategy equilibria: (Baseball,Baseball) and (Opera,Opera)
- It also has one mixed strategy equilibrium

Mixed strategies (contd.)

- Suppose the man plays the mixed strategy $(q, 1-q)$, where q is the probability he plays Baseball
- If the woman plays Baseball, her expected payoff is q
- If she plays Opera, her expected payoff is $2(1-q)$
- She is indifferent between playing Baseball and Opera iff $q = \frac{2}{3}$
- Suppose the woman plays the mixed strategy $(r, 1-r)$
- If the man plays Baseball, his expected payoff is $2r$
- If he plays Opera, his expected payoff is $1-r$
- He is indifferent between playing Baseball and Opera iff $r = \frac{1}{3}$
- The strategies $\mathbf{s_1} = (\frac{2}{3}, \frac{1}{3})$ and $\mathbf{s_2} = (\frac{1}{3}, \frac{2}{3})$ constitute a Nash equilibrium

Mixed strategies (contd.)



Mixed strategies (contd.)

- Once we allow for mixed strategies, it is not too difficult to show that *any* game with a finite number of players and finite strategy sets for each player has at least one Nash equilibrium (this is the famous Nash's theorem)
- Mixed strategies might seem a little awkward in certain setups, e.g., politics
- Later in the course we will see that we can rationalize the use of mixed strategies in an incomplete information set up. But we can already think about them from different angles

Summing up (formally)

- Given player i's (finite) pure strategy set S_i, a **mixed strategy** for player i (σ_i) assigns to each pure strategy $s_i \in S_i$ a probability $\sigma_i(s_i) \geq 0$ that it will be played, with $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$
- If $u_i(.)$ VNM utility function, payoffs are given by expected utility: $E_\sigma[u_i(s)]$
- A mixed strategy profile $(\sigma_1, ..., \sigma_I)$ is a Nash equilibrium of the game Γ_N iff for every $i = 1, ..., I$: $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$, for all $\sigma_i' \in \Delta(S_i)$, where, if we have M pure strategies, $\Delta(S_i) = \{(\sigma_{1i}, ..., \sigma_{Mi}) \in \mathbb{R}^M : \sigma_{mi} \geq 0$ for all $m = 1, ..., M$ and $\sum_{m=1}^{M} \sigma_{mi} = 1\}$
- Necessary and sufficient condition for mixed strategy σ to be Nash is that each player is indifferent among all pure strategies she plays with positive probability, *and* that these are at least as good as any pure strategy she plays with zero probability

Two (important) theorems

- Note: Allowing for mixed strategies doesn't merely involve the solution, but affects the characteristics of the game. Payoffs no longer cardinal, but expected-utility interpretation

Theorem 1 (Nash's theorem). Every game $\Gamma_N = \langle I, \Delta(S_i), u_i(.) \rangle$ in which the sets $S_1, ..., S_I$ have a finite number of elements has a mixed strategy Nash equilibrium

Theorem 2. A Nash equilibrium exists in game $\Gamma_N = \langle I, S_i, u_i(.) \rangle$ if for all $i = 1, ..., I$:

1. S_i is a nonempty, convex, and compact subset of some Euclidean space $\mathbb{R}^M$
2. $u_i(s_1, ..., s_I)$ is continuous in $(s_1, ..., s_I)$ and concave in s_i

- Note: Finite strategy set cannot be convex. In a sense, mixed strategies "convexify" strategy sets and produce well-behaved payoff functions

Dominated and rationalizable strategies

- Strategy $s_i \in S_i$ is **strictly dominated** for i if there exists $s_i' \in S_i$ such that for all $s_{-i} \in S_{-i}$: $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$
- It's **weakly dominated** if: $u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$
- If we assume all players to be rational and common knowledge of it: iterated elimination of strictly dominated strategies
- Unlike strictly dominated s, weakly dominated s cannot be ruled out based solely on principles of rationality
- In game Γ_N the strategies that survive the iterated removal of strategies that are never a best response are known as player i's **rationalizable strategies**
- Strictly dominated s is never a best response. But s might never be best response even though it's not strictly dominated
- For strictly dominated and rationalizable strategies (but not for weakly dominated), the order of removal doesn't affect the set of strategies that remain at the end of the iteration

Iterated elimination of dominated strategies

- Consider strategies 1 and 2. 1 is **strictly dominated** for a player by 2, if for each possible combination of the other players' strategies, player's payoff from playing 1 is strictly less than her payoff from playing 2

	Left	Middle	Right
Up	(1,0)	(1,2)	(0,1)
Down	(0,3)	(0,1)	(2,0)

- Right is dominated by Middle for player 2
- If we assume player 2 will not play Right, it yields the following game:

	Left	Middle
Up	(1,0)	(1,2)
Down	(0,3)	(0,1)

Iterated elimination of dominated strategies (contd.)

	Left	Middle
Up	(1,0)	(1,2)
Down	(0,3)	(0,1)

- In this game, Down is dominated by Up for player 1
- If we assume player 1 will not play Down, it yields the following game:

	Left	Middle
Up	(1,0)	(1,2)

Iterated elimination of dominated strategies (contd.)

	Left	Middle
Up	(1,0)	(1,2)

- In this game, Left is dominated by Middle for player 2
- We assume player 2 will not play Left
- This yields a predicted strategy pair: (Up,Middle)
- (Up,Middle) is an equilibrium under iterated elimination of dominated strategies

Iterated elimination of dominated strategies (contd.)

- Unfortunately, eliminating dominated strategies generally does not get us very far

	Left	Center	Right
Top	(0,4)	(4,0)	(5,3)
Middle	(4,0)	(0,4)	(5,3)
Bottom	(3,5)	(3,5)	(6,6)

- No strategies are dominated by any others, for either player
- Nash equilibrium is a solution concept that produces much tighter predictions in a broader class of games
- Let's consider a few more examples to better understand the relationships between strict dominance, weak dominance, and rationalizability ($\Rightarrow$ in sections)

What's next

- In the next class, we'll analyze rational models of electoral competition that can be studied as static games of complete information, and therefore solved by Nash equilibrium
  - Median voter theorem
  - Hotelling-Downs model of (spatial) electoral competition
  - Electoral competition with partisan candidates
  - Electoral competition with partisan candidates and uncertainty
  - Electoral competition with valence advantage