Government 2005: Formal Political Theory I

Lectures 8 & 9

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Overview

- Finitely repeated games
  - Prisoner's dilemma
  - Multiple-equilibria stage games
- Infinitely repeated games
  - Trust game
  - Prisoner's dilemma
  - Folk Theorem
- Applications
  - Cooperation between political parties
  - Collusion in Cournot duopoly (w/ Jeremy in sections)

Kick-off definitions

Definition. Given a game Γ, defined as the stage game, let $\Gamma(T)$ denote the finitely repeated game in which Γ is played T times, with the outcomes of all preceding plays observed before the next play begins. The payoffs in $\Gamma(T)$ are simply the sum of the payoffs from the T stage games.

Definition. Given a stage game Γ, let $\Gamma(\infty, \underline{\delta})$ denote the infinitely repeated game in which Γ is repeated forever and the N players of Γ have discount factors $\underline{\delta} = (\delta_1, ..., \delta_N)$. For each play t, the outcomes of the $(t-1)$ preceding plays of the stage game are observed before stage t begins. Each player's payoff in $\Gamma(\infty, \underline{\delta})$ is the present value of the player's payoffs from the infinite sequence of the stage games.

Finitely repeated games with unique NE in the stage game

- In repeated games, players play the same "stage game" repeatedly, and observe the moves played in previous stage games as they play
- We first consider the case of **finitely** repeated games
- In finitely repeated games, if there is a **unique** Nash equilibrium to the stage game, then the **unique** subgame perfect equilibrium in the repeated game is for the players to play the Nash equilibrium of the stage game in every period
- We can simply show this by backward induction in the case of the prisoner's dilemma

Finitely repeated prisoner's dilemma

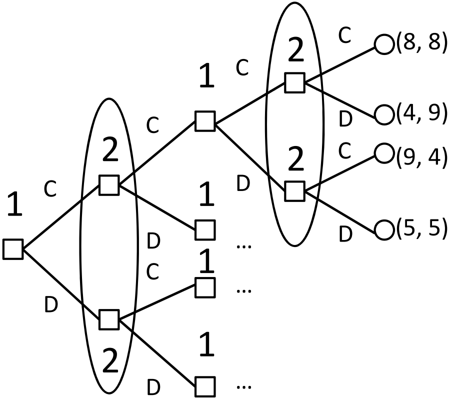
- To see this, consider as the stage game the prisoners' dilemma with the following payoffs:

	D	C
D	(1,1)	(5,0)
C	(0,5)	(4,4)

- There is a unique Nash equilibrium of the stage game: (D,D)

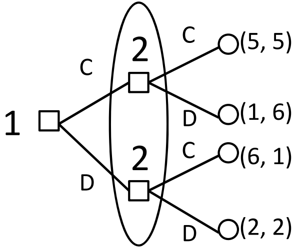
Finitely repeated prisoner's dilemma (contd.)

- If this game is repeated twice ($T = 2$), the extensive form is:



Finitely repeated prisoner's dilemma (contd.)

- Regardless of what happens in the first period, players 1 and 2 will both defect in the second period
- Thus, players know that (C,C) in the first period leads to the payoff $(5, 5)$
- Similar calculations for the other branches yield the following payoffs at the first period:



Finitely repeated prisoner's dilemma (contd.)

- Basically, we are adding the payoffs of (D,D) in the second period to the first period strategies
- The first-period strategic interaction is thus captured by the following (fictitious) strategic representation:

	D	C
D	(2,2)	(6,1)
C	(1,6)	(5,5)

Finitely repeated prisoner's dilemma (contd.)

- Both players will defect in the first period as well
- This logic clearly generalizes to any finite number of periods T, as the game "unravels from the end"
- And the result is clearly more general (i.e., it goes beyond the repeated prisoner's dilemma)

Theorem. If the stage game Γ has a unique NE, then, for any finite T, the repeated game $\Gamma(T)$ has a unique SPNE: The NE of Γ is played in every stage.

Multiple-equilibria stage games

- If there are multiple NE in the stage game, the situation is more interesting (i.e., let the cherry-picking begin)
- Consider a two-period game with the following stage game (see Gibbons):

	D	C	R
D	(1,1)	(5,0)	(0,0)
C	(0,5)	(4,4)	(0,0)
R	(0,0)	(0,0)	(3,3)

- There are two Nash equilibria: (D, D) and (R, R)

Multiple-equilibria stage games (contd.)

- Players may *believe* that different outcomes will happen in the second period depending on what happens in the first period
- Our goal is to construct a subgame perfect equilibrium where there is cooperation (after all, we are "benevolent" cherry-pickers...)
- How can we do this?
- We know that we cannot achieve this in the second stage of the game (backward induction)
- Then, the idea is to *play with the beliefs* about the NE players will play in the second stage of the game to induce cooperation in the first one

Multiple-equilibria stage games (contd.)

- Suppose players expect the NE (R, R) to be played in the second period if the players play (C, C) in the first period
- However, they expect the NE (D, D) to be played if either player plays D or R in the first period
- Then, at the first period the anticipated payoffs are:

	D	C	R
D	(2,2)	(6,1)	(1,1)
C	(1,6)	(7,7)	(1,1)
R	(1,1)	(1,1)	(4,4)

- There are three Nash equilibria of this game with the mentioned beliefs: (C, C), (D, D), (R, R)

Multiple-equilibria stage games (contd.)

- Thus, there are three subgame perfect equilibria in the repeated game with the mentioned beliefs: $((C, C), (R, R))$, $((D, D), (D, D))$, and $((R, R), (D, D))$
- Cooperation in the first period is sustainable as an equilibrium
- **Potential problem:** *Why* should the players expect the (D, D) equilibrium ever to happen in the second period?
  - This outcome is *Pareto inferior* to the (R, R) equilibrium
- Playing with beliefs, you can get many **multiple equilibria**
  - Any combination of the Nash equilibria of the stage game are also subgame perfect equilibria

Multiple-equilibria stage games (contd.)

- Consider a two-period game with the following stage game (see, again, Gibbons):

	D	C	R	P	Q
D	(1,1)	(5,0)	(0,0)	(0,0)	(0,0)
C	(0,5)	(4,4)	(0,0)	(0,0)	(0,0)
R	(0,0)	(0,0)	(3,3)	(0,0)	(0,0)
P	(0,0)	(0,0)	(0,0)	(4,1/2)	(0,0)
Q	(0,0)	(0,0)	(0,0)	(0,0)	(1/2,4)

- There are 4 Nash equilibria: (D, D), (R, R), (P, P), (Q, Q)
- (R, R) Pareto dominates (D, D)
- But (R, R), (P, P), and (Q, Q) are not Pareto dominated by any other equilibrium

Multiple-equilibria stage games (contd.)

- Assume the following beliefs about what NE will prevail in the second stage based on the outcome of the first stage:
  - (R, R) if (C, C)
  - (P, P) if (C, w), with $w \neq C$
  - (Q, Q) if (x, C), with $x \neq C$
  - (R, R) if (y, z), with $y, z \neq C$

	D	C	R	P	Q
D	(4,4)	(5.5,4)	(3,3)	(3,3)	(3,3)
C	(4,5.5)	(7,7)	(4,0.5)	(4,0.5)	(4,0.5)
R	(3,3)	(0.5,4)	(6,6)	(3,3)	(3,3)
P	(3,3)	(0.5,4)	(3,3)	(7,3.5)	(3,3)
Q	(3,3)	(0.5,4)	(3,3)	(3,3)	(3.5,7)

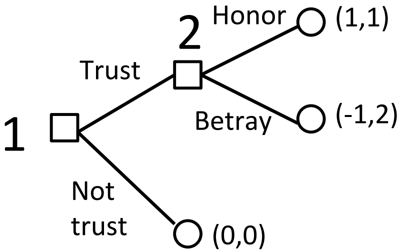
- The (fictitious) first-stage game has three Nash equilibria: (D, D), (C, C), (R, R)

Multiple-equilibria stage games (contd.)

- Again, we have shown that cooperation is sustainable in the subgame perfect equilibrium under an appropriate set of beliefs
- And now the players don't need to use a Pareto dominated outcome as punishment device (as it was instead the case in the previous example)
- Each player has a Pareto undominated outcome which can be "used" to punish the other player
- That's why, unlike the previous example, if punishment is called, the punisher doesn't want to renegotiate (loosely speaking)

Infinitely repeated trust game

- Consider the following stage (trust) game:



- The stage game has a unique SPNE with outcome $(0, 0)$ (which is also the unique NE)
- We assume that it's repeated infinitely with $\delta_1 = \delta_2 = \delta$

Infinitely repeated trust game (contd.)

- Define the **grim trigger strategy** profile for the players as:
  - Player 1 plays T in the first period. Thereafter, she plays T if in all previous periods all plays have been T and H; she plays N otherwise
  - Player 2 plays H (if given to act) if in all previous periods all plays have been T and H; she plays B otherwise
- Under the threat of trigger strategies, **cooperation is sustainable** (i.e., (T, H) is an equilibrium outcome resulting from the above Nash strategy profile) iff $\delta \geq 1/2$
  - In fact, player 2 has no incentive to defect from cooperation if $\frac{1}{1-\delta} \geq 2 \Rightarrow \delta \geq 1/2$
  - Player 1 has no incentive to defect from cooperation, otherwise she gets zero forever (she cannot cheat here)

Infinitely repeated trust game (contd.)

- More generally, if we define C as the one-stage payoff from cooperation, D from defection, and P from punishment, with $D > C > P$, player 2 sticks to cooperation iff

$$\frac{C}{1-\delta} \geq D + \frac{\delta P}{1-\delta}$$

$$\delta \geq \frac{D-C}{D-P} = K$$

- If C decreases and/or P increases and/or D increases, then K increases and cooperation is harder to sustain as equilibrium outcome

Infinitely repeated prisoner's dilemma

- Suppose the prisoner's dilemma from before is repeated infinitely and payoffs are discounted by a discount factor $\delta < 1$

	D	C
D	(1,1)	(5,0)
C	(0,5)	(4,4)

- Since there is no "last period" we cannot solve the game backwards from the end
- Instead, we postulate a pair of strategies, then check whether these strategies constitute an equilibrium
- As before, an equilibrium is subgame perfect iff the strategies are a Nash equilibrium in *all* subgames
  - In the repeated-game context, each period begins a subgame
  - Be careful not to confuse stage games with subgames

Infinitely repeated prisoner's dilemma (contd.)

- **In infinitely repeated games, if players are patient enough, there are *tons* of subgame perfect equilibria**
- We start by postulating one of the possible equilibria in which both players adopt *grim trigger strategies*
- In the first period, player i plays C. In period t, if the outcomes of all preceding stages $(t-1)$ have been (C, C), she plays C; otherwise she plays D forever after
- Is cooperation sustainable as an equilibrium outcome with these (equilibrium) strategies? Is it NE? Is it SPNE?

Infinitely repeated prisoner's dilemma (contd.)

- Sustained cooperation delivers $\rightarrow \frac{4}{1-\delta}$
- Deviating from cooperation delivers $\rightarrow 5 + \frac{\delta}{1-\delta}$
- The former quantity is greater than the latter iff $\delta \geq \frac{1}{4}$
- Note: Playing D in every period is NE (and also SPNE). Thus, the punishment strategy is credible
- If players are patient enough, *i.e.*, as long as $\delta \geq \frac{1}{4}$, then cooperating every period is a NE outcome (sustained by the off-equilibrium punishment of grim trigger strategies)

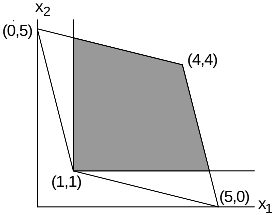
Infinitely repeated prisoner's dilemma (contd.)

- Are these strategies also SPNE?
- There are two types of subgames:
  (i) All preceding outcomes are (C,C)
  (ii) At least one preceding outcome is different from (C,C)
- First is like original game, so trigger strategies are NE there too as long as $\delta \geq \frac{1}{4}$
- Second is also infinitely repeated prisoner's dilemma, for which (D,D) forever is NE
- Therefore, grim trigger strategies equilibrium is subgame perfect iff $\delta \geq \frac{1}{4}$
- It can also be shown by single-deviation principle
- Note: Trigger strategies where first player only punishes (.,D) as opposed to (D,D) is NE but not SPNE

Infinitely repeated prisoner's dilemma (contd.)

Folk Theorem

- *Almost any* feasible and individually rational outcome can be sustained as a subgame perfect equilibrium
- For the prisoners' dilemma above, the payoffs sustainable as subgame perfect equilibria are:



- The set of feasible outcomes is the set bounded by the points $(0,5)$, $(1,1)$, $(5,0)$ and $(4,4)$
- The set of "individually rational" payoffs are those with $x_1 > 1$ and $x_2 > 1$

Infinitely repeated prisoner's dilemma (contd.)

Folk Theorem (contd.)

- Define π as the discounted average of the stream $(\pi_1, \pi_2, \ldots)$

$$\pi = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \pi_t$$

- Define $\underline{v}_i$ as the minimax value of player i

$$\underline{v}_i = min_{s_{-i}} \left[max_{s_i} u_i(s_i, s_{-i}) \right]$$

- Note: If δ close to 1, all weighted averages of stage-game payoffs are feasible as average payoffs of $\Gamma(\infty, \delta)$

Folk Theorem. For every feasible average payoff vector v such that $v_i > \underline{v}_i$ for all i, there exists a $\delta' < 1$ such that for all $\delta \in (\delta', 1)$ there is a NE of $\Gamma(\infty, \delta)$ with payoffs v.

Infinitely repeated prisoner's dilemma (contd.)

- The use of grim trigger strategies raises the question of *renegotiation* (loosely speaking)
- Off the equilibrium path, players are stuck in a phase of "punishment forever"
- It seems plausible that they would want to renegotiate to a better outcome
- The problem is that, if players expect such renegotiation to occur, then the *initial threat* to move to a "punishment phase" becomes weaker
- Let's postulate another possible equilibrium with a "shorter" punishment phase

Infinitely repeated prisoner's dilemma (contd.)

Tit-for-tat

- **Tit-for-tat** strategies: Cooperate in the first stage and then in any subsequent stage use the action that the other player chose in the previous stage
- If C forever $\rightarrow 4/(1-\delta)$
- If D forever $\rightarrow 5+\delta/(1-\delta)$
- If D and then back to tit-for-tat $\rightarrow 5/(1-\delta^2)$
- Hence, for tit-for-tat to be NE, both conditions must hold:

$$\frac{4}{1-\delta} \geq 5+\frac{\delta}{1-\delta} \Rightarrow \delta \geq \frac{1}{4}$$

$$\frac{4}{1-\delta} \geq \frac{5}{1-\delta^2} \Rightarrow \delta \geq \frac{1}{4}$$

Infinitely repeated prisoner's dilemma (contd.)
Tit-for-tat (contd.)

- But is it also SPNE?
- By single-deviation principle, in the subgames following unilateral defection, player *i* (the punisher) needs to prefer oscillation to cooperation forever in order to carry out the punishment $\Rightarrow \delta \leq \frac{1}{4}$
- In the subgames following cooperation in every preceding period, player *i* needs to prefer cooperation forever to oscillation $\Rightarrow \delta \geq \frac{1}{4}$
- Therefore, SPNE iff $\delta = \frac{1}{4}$ (very razor-edge condition)

- Between the infinite and the one-period punishment, there are many *intermediate* punishment strategies that, under certain parameter conditions, can sustain cooperation in equilibrium

Cooperation between political parties

- Suppose two parties compete for control of government
- There is a sequence of elections, and after each election the winning party implements a policy
- The parties have policy preferences over the policy space $[0, 1]$
- Party 1's preferences are given by $u_1(x_t) = -x_t^2$, and party 2's preferences are given by $u_2(x_t) = -(x_t - 1)^2$, where x_t is the policy enacted in period t
- Party 1's ideal point is thus at $z_1 = 0$ and party 2's ideal point at $z_2 = 1$
- The parties discount future payoffs using a common discount factor, δ

Cooperation between political parties (contd.)

- In each election it is uncertain which party will win
- Assume the probability that party 1 wins each election is fixed at $p \in (0, 1)$
- The **stage game** is simple: nature chooses a winning party, and that party then chooses a policy
- In equilibrium, the party that wins simply implements its ideal point, since this is a dominant strategy
- If the one-shot equilibrium is repeated over and over, then policy flips back and forth between 0 and 1; the **expected value** is $p[0] + (1 - p)[1] = (1 - p)$

Cooperation between political parties (contd.)

- Because the parties are risk-averse, they would rather cooperate and have more policy stability
- For example, both parties would be better off if the expected policy $1-p$ was enacted in every period
- To see this, let $V_i(x_1, x_2)$ be party i's total discounted expected payoff when party 1 enacts the policy x_1 in every period and party 2 enacts the policy x_2 in every period
- Then, the non-cooperative equilibrium payoff for party 1 is

$$\begin{aligned} V_1(0,1) &= -\sum_{t=0}^{\infty} \delta^t [p(0) + (1-p)(1)] \\ &= -\frac{1-p}{1-\delta} \end{aligned}$$

Cooperation between political parties (contd.)

- The cooperative equilibrium payoff for party 1 is

$$\begin{aligned} V_1(1-p, 1-p) &= -\sum_{t=0}^{\infty} \delta^t [p(1-p)^2 + (1-p)(1-p)^2] \\ &= -\frac{(1-p)^2}{1-\delta} \end{aligned}$$

- Since $p \in (0,1)$, $V_1(1-p, 1-p) > V_1(0,1)$
- The analogous holds for party 2

Cooperation between political parties (contd.)

- Can the parties achieve a **cooperative equilibrium** in the infinitely repeated game?
- Consider the strategies: "Play $x = 1 - p$ as long as the other party has played $1 - p$ every time it has been in power; otherwise, play your ideal point forever"
- If party 1 is in power in some period, then if it plays the cooperative strategy its total discounted expected payoff is $V_1(1 - p, 1 - p)$
- If party 1 deviates by enacting a policy $x \neq 1 - p$, then it knows that party 2 will play $x_2 = 1$ in all future periods when party 2 is in power
- Party 1 also knows that it will play $x_1 = 0$ in all future periods when it is in power

Cooperation between political parties (contd.)

- Party 1's total discounted expected payoff from that point on is then
$$V_1^D = 0 + \delta V_1(0,1) = -\frac{\delta(1-p)}{1-\delta}$$
- Party 1 has no incentive to deviate if V_1^D is less than or equal to $V_1(1-p, 1-p)$, that is, if
$$\begin{aligned} -\frac{(1-p)^2}{1-\delta} &\geq -\frac{\delta(1-p)}{1-\delta} \\ \delta &\geq 1-p \end{aligned}$$
- Similar calculations for party 2 show that it has no incentive to deviate if $\delta \geq p$
- The cooperative equilibrium is subgame perfect iff $\delta \geq max\{p, 1-p\}$

Cooperation between political parties (contd.)

- This cooperative equilibrium is easier to sustain when the parties have approximately equal chances of winning each election, *i.e.*, when $p \approx \frac{1}{2}$
- When this cooperative equilibrium cannot be sustained, the problem lies with the *weaker party*
- This is intuitive: If $p < \frac{1}{2}$, then party 1 does not expect to be in power very often, and the "compromise" in the cooperative equilibrium puts the policy closer to party 2's ideal point than party 1's ideal point
- So, when party 1 does happen to get into power, it is much more tempted to deviate than party 2 is

Cooperation between political parties (contd.)

Cooperation implementing the "fair" outcome

- Consider now the "fair" outcome: "play $x = \frac{1}{2}$ each period"
- The total discounted expected payoffs under these strategies are the same for both parties:

$$V_1(\tfrac{1}{2}, \tfrac{1}{2}) = V_1(\tfrac{1}{2}, \tfrac{1}{2}) = -\sum_{t=0}^{\infty} \delta^t [p(\tfrac{1}{2})^2 + (1-p)(\tfrac{1}{2})^2] = -\frac{1}{4(1-\delta)}$$

- Under the "grim trigger" strategies described above, party 1 has no incentive to deviate if

$$\begin{aligned} -\frac{1}{4(1-\delta)} &\geq -\frac{\delta(1-p)}{1-\delta} \\ \delta &\geq \frac{1}{4(1-p)} \end{aligned}$$

Cooperation between political parties (contd.)

- Similarly, party 2 has no incentive to deviate if

$$
\begin{aligned}
-\frac{1}{4(1-\delta)} &\geq -\frac{\delta p}{1-\delta} \\
\delta &\geq \frac{1}{4p}
\end{aligned}
$$

- This cooperative equilibrium is easier to sustain when $p \approx \frac{1}{2}$
- When the cooperative equilibrium cannot be sustained, the problem lies with the *stronger party*
- If $p > \frac{1}{2}$, the cooperative outcome produces a stream of policies with a lower variance than the non-cooperative outcome, which party 1 likes
- But the cooperative outcome produces a stream of outcomes with a mean *further* from party 1's ideal point than the mean of the non-cooperative outcome, which party 1 dislikes

Cooperation between political parties (contd.)

Limited cooperation

- What is the maximal amount of (stationary) cooperation sustainable when "full" cooperation is not sustainable?
- Consider the symmetric case, with $p = \frac{1}{2}$
- Our analysis so far shows that full cooperation is possible, with $x_1 = x_2 = \frac{1}{2}$ in all periods, if $\delta \geq \frac{1}{2}$
- We want to know what is the minimal amount of divergence sustainable as a subgame perfect equilibrium if $\delta < \frac{1}{2}$
- Consider the strategies "party 1 plays $x_1 = \frac{1}{2} - \theta$ in all periods" and "party 2 plays $x_2 = \frac{1}{2} + \theta$ in all periods," where $0 \leq \theta \leq 1/2$
- θ measures the *extent of the deviation from cooperation*

Cooperation between political parties (contd.)

- Let $\tilde{V}_1(x_1, x_2)$ be party 1's total discounted expected payoff when party 1 enacts the policy x_1 in every period, party 2 enacts the policy x_2 in every period, <u>and</u> party 1 is currently in power:

$$\begin{aligned}
\tilde{V}_1(x_1, x_2) &= -x_1^2 - \sum_{t=1}^{\infty} \delta^t (\tfrac{1}{2})[x_1^2 + x_2^2] \\
&= -x_1^2 - \tfrac{\delta}{2(1-\delta)}[x_1^2 + x_2^2]
\end{aligned}$$

- Similarly, when party 2 is currently in power:

$$\begin{aligned}
\tilde{V}_2(x_1, x_2) &= -(1-x_2)^2 - \sum_{t=1}^{\infty} \delta^t (\tfrac{1}{2})[(1-x_1)^2 + (1-x_2)^2] \\
&= -(1-x_2)^2 - \tfrac{\delta}{2(1-\delta)}[(1-x_1)^2 + (1-x_2)^2]
\end{aligned}$$

Cooperation between political parties (contd.)

- Substituting the (limited) cooperative strategies $(x_1, x_2) = (\frac{1}{2} - \theta, \frac{1}{2} + \theta)$ into $\tilde{V}_1$, $\tilde{V}_1$ becomes

$$\tilde{V}_1(\tfrac{1}{2} - \theta, \tfrac{1}{2} + \theta) = -(\tfrac{1}{2} - \theta)^2 - \tfrac{\delta}{1-\delta}[\tfrac{1}{4} + \theta^2]$$

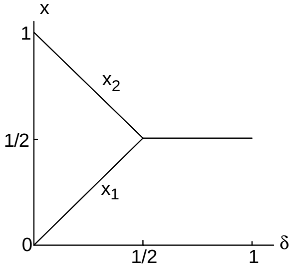
- If party 1 deviates, its total discounted expected payoff is $V_1^D = -\frac{\delta}{2(1-\delta)}$
- So, party 1 has no incentive to deviate iff

$$\begin{aligned}
-(\tfrac{1}{2} - \theta)^2 - \tfrac{\delta}{1-\delta}[\tfrac{1}{4} + \theta^2] &\geq -\tfrac{\delta}{2(1-\delta)} \\
\tfrac{\delta}{1-\delta}[\tfrac{1}{4} - \theta^2] &\geq (\tfrac{1}{2} - \theta)^2 \\
\tfrac{\delta}{1-\delta}(\tfrac{1}{2} - \theta)(\tfrac{1}{2} + \theta) &\geq (\tfrac{1}{2} - \theta)^2 \\
\theta &\geq \tfrac{1}{2} - \delta
\end{aligned}$$

- Analogous calculations show that the same inequality must hold for party 2

Cooperation between political parties (contd.)

- So, the maximal amount of "cooperation" – *i.e.*, the minimal amount of divergence – sustainable is given by $\theta_{MIN} = \frac{1}{2} - \delta$
- When $\delta = 0$ (no patience) no cooperation is possible, and as δ rises towards $\frac{1}{2}$ cooperation increases in a smooth (in fact, linear) fashion
- The maximally cooperative strategies are shown in the following figure



Where are we?

- We have studied **dynamic games of complete information** (including **repeated games**)
- References:
  - Lecture slides $\rightarrow$ 5 through 9 (final folder)
  - Osborne $\rightarrow$ chapters 5 through 7 + 14 + 16.1
  - Gibbons $\rightarrow$ chapter 2
  - McCarty & Meirowitz $\rightarrow$ chapters 7 + 9 + 10.2
- Now we'll move on to:
  - Static games of incomplete information (lecture 10)
  - Dynamic games of incomplete information (lectures 11-12)